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Quantum reparametrizations in the two-dimensional gravity: a look from $2+\epsilon$ dimensions

S. NAFTULIN ^{*}

*Institute for Single Crystals,
60 Lenin Ave., 310141 Kharkov, Ukraine*

Abstract

We discuss the structure of one-loop counterterms for the two-dimensional theory of gravitation in the covariant scheme and study the effect of quantum reparametrizations. Some of them are shown to be equivalent to the introduction of $2 + \epsilon$ -dimensional terms into the initially 2-dimensional theory. We also argue that the β -function for the Einstein constant has a non-trivial ultraviolet stable point beyond two dimensions.

^{*} Electronic address: naftulin@isc.kharkov.ua

1 Introduction

The celebrated ϵ -expansion devised primarily for the needs of critical phenomena (see [1] and references therein) has paved its way into the quantum theory of gravitation [2, 3]. The approach received a new impact when it was realized that it pays to start from $d = 2 + \epsilon$ and then analytically continue $\epsilon \rightarrow 2$. (The odd-dimensional theories are generally excluded from such an analytic continuation.)

Basically, some of geometrical terms may drop out as the space-time dimension decreases by an integer. However, at non-integer values of d there is no independent Lagrangian construction, and the theory is defined by an analytic continuation in the parameter space. Thus one can expect that a smooth reduction of d (in the spirit of the renormalization group) leaves its “fingerprints”, in the reparametrization structure of the resulting theory: the dropped structures degenerate and mix with those left. An obvious example is two actions quadratic in curvature, $S_1 = \int d^2x \sqrt{g} R^{\mu\nu} R_{\mu\nu}$ and $S_2 = \frac{1}{2} \int d^2x \sqrt{g} R^2$, which become indistinguishable in two dimensions, by the Bianchi identities:

$$R_{\mu\nu} = \frac{1}{2} R g_{\mu\nu} . \quad (1)$$

The first impulse is to discard S_1 as we descend to a $d = 2$ theory, but this would simply entail the impossibility of returning to higher dimensions. A more sophisticated option is to allow for an operator mixing; the structure degenerates as we approach $d = 2$ and the degeneracy should be reflected in the invariance of the resulting theory with respect to possible parametrizations. Conversely, as we perform an analytic continuation beyond $d = 2$ the degeneracy, (1), is lifted and additional terms come into effect. Hence the study of different quantum parametrizations is not completely meaningless,¹ although the on-shell effective actions are parametrization independent up to topologically trivial surface terms [4, 5].

¹We do not touch upon reparametrization anomalies here.

Another relevant problem is the presence of ultraviolet divergences in any integer space-time dimension. There are several covariant regularizations, of which the most suitable for our purposes is the dimensional reduction: $d = 2 \rightarrow 2(1-\epsilon)$, $\epsilon \geq 0$. Although there are no singularities for intermediate values of ϵ , the limit $\epsilon \rightarrow 0$ is special at each order of the perturbation theory in the Einstein constant κ . For small but finite values of ϵ a vast number of terms arises in the effective action; however, the would-be convergent terms contribute an extra power of ϵ , which is negligibly small within the loop expansion. We will restrict our attention to the one-loop approximation and small ϵ . Consequently the quantum corrections which are finite in the limit $\epsilon \rightarrow 0$, will be of no interest to us.

It has been known for some time that the treatment of the $d = 2$ Einstein theory based upon the dimensional regularization is afflicted by oversubtractions, [3], since the action $\int d^2x \sqrt{g}R$ is a topological invariant. Due to peculiar features of the conformal mode at $\epsilon \rightarrow 0$ there seems to be a better candidate for a $2 + \epsilon$ -dimensional theory of gravitation, viz., the Jackiw-Teitelboim model [6]. It is very likely that this model does not admit the problem of oversubtractions at $\epsilon \rightarrow 0$ because the dilaton, Φ , field may be viewed as arising from the integration over the $d = 2$ conformal anomaly, [7]. The interplay between the dilaton and the conformal mode should result in a cancellation of the kinematical ϵ^{-1} -poles in the propagators.

The dimensional regularization scheme has another important advantage: it automatically eliminates the contributions like $\delta(0)$ to the effective action. Due to this property the dimensional regularization ensures consistency of the naive determinant calculations, which would otherwise require a modification when applied to gravity [8].

At the first sight, the conformal properties of two-dimensional models of gravity suggest the natural, conformal, parametrization of the quantum metric fluctuations $g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = (\exp \sigma) g_{\mu\nu}$ (we use the language of the background field method [5] throughout the paper, also see a book [9] for a comprehensive review). But on the other hand, the conformal representation

does not seem very suitable in view of a possible extension to the higher-dimensional world ($\epsilon \rightarrow 2$). For example, the conformal approach to the $d = 4$ gravity [10] has attained a limited success as the nice properties of exact solvability are destroyed. Thus it may be crucial to see what the picture looks like [11] in the conventional, linear, representation $g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$.

In this paper, we re-examine the structure of one-loop counterterms in two-dimensional gravities and make a brief glance at how the reparametrization effects modify the considerations. Although the main logical emphasis might be on the conventional gravity, we find it more convenient to start (Section 2) with its higher-order counterpart. Section 3 contains a paralleling treatment of the dilaton gravity, and a short Section 4 is devoted to conclusions.

2 Resurrecting Gravitons in the R^2 -gravity

The two-dimensional R^2 quantum gravity does not apparently have any independent physical significance and mainly serves as a good playground. It was extensively studied within the Arnowitt-Deser-Misner formalism and was found to have zero propagating degrees of freedom [12] and hence a trivial S -matrix.² Of course, the latter does not imply that the model describes trivial space-time manifolds, nor that its ultraviolet divergences are absent.

Consider the linear background versus quantum metric splitting $g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$ and further decompose the quantum fluctuation into its trace $h = g^{\mu\nu}h_{\mu\nu}$ and the traceless “transverse graviton” $\bar{h}_{\mu\nu} = h_{\mu\nu} - (1/2)hg_{\mu\nu}$. Since there are no transverse directions in the (1+1)-dimensional space-time, it is natural to assume that $\bar{h}_{\mu\nu}$ is just a gauge artifact; and indeed, one can adopt the gauge $\bar{h}_{\mu\nu} = 0$, which is essentially equivalent to the background conformal gauge $h_{\mu\nu} = (e^\sigma - 1)g_{\mu\nu}$.

In the fourth-order theory, the number of degrees of freedom is effectively

²The same feature is shared by the Jackiw-Teitelboim model and its generalizations.

doubled [13] as compared to the conventional (second-order) one, so we can expect that beside h another dynamical scalar exists. This is at the heart of the approach suggested by Yoneya [12]: using an auxiliary scalar field Φ the fourth order Lagrangian

$$S = - \int d^2x \sqrt{g} (\omega R^2 + \Lambda) \quad (2)$$

may be represented as

$$S = - \int d^2x \sqrt{g} \left(R\Phi - \frac{1}{4\omega} \Phi^2 + \Lambda \right). \quad (3)$$

In the above definition, (2), we have neglected a topological term

$$-\frac{1}{2\kappa^2} \int d^2x \sqrt{g} R, \quad (4)$$

because its appearance does not affect the divergent structure and only leads to a constant shift of the “dilaton” field: $\Phi \rightarrow \Phi + (1/2\kappa^2)$.

Thus the problem of finding the one-loop structure of counterterms reduces to that for dilaton gravities. The latter can be solved by a variety of methods, either conformal or covariant. In the background field formulation, the divergent contribution to the one-loop effective action, Γ , is, [14]:

$$\Gamma_{div} = -\frac{1}{4\pi\epsilon} \int d^2x \sqrt{g} R \quad (5)$$

plus curvature terms at the one-dimensional boundary; here $\epsilon = (2-d)/2$ is the dimensional regulator.

The above result is not totally unexpected: Eq.(5) is the only metric-covariant local expression with the appropriate background dimensionality. Since $\int d^2x \sqrt{g} R$ is a topological invariant Eq.(5) is consistent with the anticipation that the S -matrix is finite. To summarize, the model (2) is renormalizable in the generalized sense (i.e., after inclusion of the topological term (4) into the bare action), the cosmological constant, Λ , remains finite.

Thus far, the treatment has been straightforward and it is not at all evident why bother studying the same model in the covariant scheme. (One

such motivation might be to include the term $\int d^2x \sqrt{g} R^{\mu\nu} R_{\mu\nu}$ into the bare action (2) in the vicinity of $d = 2$, see below). Even the first step of the evaluation brings about surprises, in the guise of technical obstacles. Thus it is worth while to say a few words of the Schwinger-DeWitt technique (see, e.g., [8] for a general review, and the second paper in [11] for applications to the two-dimensional gravity.)

The whole procedure reduces to the calculation of the determinant of the fourth-order differential operator \widehat{H}_{ij} , which is essentially the second functional derivative of the action (2). This operator contains 2×2 -matrices acting in the space of the quantum fields $\{h; \bar{h}_{\mu\nu}\}$. If minimal gauge conditions are used, it takes the form

$$\widehat{H}_{ij} = -\widehat{K}_{ij}\Delta^2 + \text{lower-order derivatives .} \quad (6)$$

The divergences are defined by the coincidence limits in the heat kernel expansion, or equivalently, through the “universal functional traces” [8] obtained by iterating Eq.(6) with respect to its highest-order term $\widehat{K}_{ij}\Delta^2$. Congenially, the symmetric matrix \widehat{K}_{ij} is usually taken to be a metric in the configuration space of the quantum fields and hence defines the quantum measure in the path integral, [15].

Quite curiously, expanding the basic action (2) in powers of the quantum fields one verifies that there appears no term like $\bar{h}^{\mu\nu}\Delta^2\bar{h}_{\mu\nu}$ so that the matrix \widehat{K}_{ij} in Eq.(6) is degenerate, thus reminding that the space of states is full of gauge phantoms—and that the field $\bar{h}_{\mu\nu}$ is just one of them. Indeed, in higher dimensions the corresponding term comes from the Weyl tensor squared (see, e.g., [13]), which identically vanishes for $d = 2$. The subsequent analysis may be twofold: one could either gauge the field $\bar{h}_{\mu\nu}$ away adopting the conformal parametrization, or rather invent some way to change the relative weight of the conformal mode in the $h_{\mu\nu}$ -loop by hand. It can be argued [16] that the both procedures may be fitted so as to give equivalent off-shell expressions for the one-loop divergences (up to surface terms). Here we only take the second option.

In order to modify the conformal mode at the quantum level without breaking the general co-ordinate covariance, let us consider the following term:³

$$\delta S = -\xi \omega \int d^2x \sqrt{\bar{g}} \bar{h}^{\mu\nu} \Delta \left[\tilde{R}_{\mu\nu} - \frac{1}{2} \tilde{R} \tilde{g}_{\mu\nu} \right]. \quad (7)$$

Here the quantities with tildes contain both background and quantum components and hence must be Taylor expanded (to the first order in fluctuations). The weight factor ξ is arbitrary: as the expression in the square brackets on the right-hand side of Eq.(7) is zero at exactly $d = 2$ by the virtue of the Bianchi identities (1), the divergent part of the effective action obviously does not depend on ξ .⁴

Adding Eq.(7) to the initial action (2) is equivalent to a reparametrization of the quantum fields so that the divergences are not affected. However, one cannot send ξ to zero until the evaluation is complete because the matrix \widehat{K}^{-1} gets diverged. Further, the total contribution to the path integral measure, $(i/2) \log \det(-\widehat{K}_{ij})$, is proportional to $\delta(0) \times \log \xi$, that is why the use of the dimensional regularization is preferable: then $\delta(0)$ is regulated to zero. The intermediate expressions acquiring a pole at $\xi = -1$, we confine ourselves to the domain $\xi > 0$.

Unfortunately, it is not evident either if the ghost operator becomes both minimal and non-degenerate for $\xi \neq 0$: this pivotal point must be checked explicitly. As typical in the higher-order gauge theories, the gauge fixing action,

$$S_{g.f.} = - \int d^2x \sqrt{g} \chi^\mu \widehat{C}_{\mu\nu} \chi^\nu, \quad (8)$$

contains the operator-valued $\widehat{C}_{\mu\nu}$ so that the third ghost should be accounted for. A relatively simple choice of gauge is:

$$\chi^\mu = -\nabla_\nu \bar{h}^{\mu\nu} + \frac{1}{2(1+\xi)} \nabla^\mu h, \quad (9)$$

³To our knowledge, a similar term was firstly introduced in Ref.[17] in a somewhat different setting.

⁴Rigorously, renormalization of the background fields might be required to eliminate the reference to ξ in the effective action.

$$\hat{C}_{\mu\nu} = \xi g_{\mu\nu} \Delta + \nabla_\mu \nabla_\nu - \xi R_{\mu\nu} . \quad (10)$$

The total contribution to the one-loop effective action is given by the standard expression

$$\Gamma = \frac{i}{2} \text{Tr} \log \hat{H} - i \text{Tr} \log \hat{\mathcal{M}} + \frac{i}{2} \text{Tr} \log \hat{C} , \quad (11)$$

where the first term is determined by the quadratic expansion of Eqs.(2), (7):

$$\hat{H}_{ij} = \left(S^{(2)} + \delta S^{(2)} + S_{g.f.} \right)_{ij} , \quad (12)$$

the second term is the Faddeev-Popov ghost operator:

$$\begin{aligned} \hat{\mathcal{M}}_{\mu\nu} &\equiv \hat{C}_{\mu\lambda} \frac{\delta \chi^\lambda}{\delta \omega^\nu} \\ &= \xi \Delta^2 g_{\mu\nu} + R \nabla_\mu \nabla_\nu + \text{lower-order derivatives} , \end{aligned} \quad (13)$$

and the third one is due to Eq.(10). Infinitesimal parameters of the gauge transformations in Eq.(13) are denoted by ω^ν .

The last term in Eq.(11) coincides (up to the contribution to the functional measure) with the one-loop QED determinant in the background Lorentz gauge with the parameter $\lambda = -1/(1 + \xi)$. In two space-time dimensions the divergent part of the QED is a λ -independent [8] surface term, though it is well-defined only for $\lambda > -1$, i.e., for $\xi > 0$.

With the account of the gauge fixing term (8), the ξ -dependence penetrates into the denominators of the configuration space matrices in \hat{H}_{ij} . The Vilkovisky-DeWitt metric with the “resurrected” graviton becomes

$$\hat{K} = \frac{\omega}{2} \begin{pmatrix} \xi/(1 + \xi) & 0 \\ 0 & -\xi (g^{\mu\alpha} g^{\nu\beta} + g^{\mu\beta} g^{\nu\alpha} - g^{\mu\nu} g^{\alpha\beta}) \end{pmatrix} . \quad (14)$$

Note that the hh -propagator also gets modified, viz., $1 \rightarrow \xi/(1 + \xi)$ in its numerator. Now there is a simple way to see how the $\bar{h}_{\mu\nu} \bar{h}_{\alpha\beta}$ -sector decouples: set $\xi \rightarrow \infty$ for instance.

In Ref.[16] we have demonstrated that all the ξ -dependent terms cancel exactly from the final expression for Γ_{div} . This is a pleasant surprise since

in a general case renormalization of the metric $g_{\mu\nu}$ is needed to eliminate ξ from the effective action. We conclude that the divergences calculated in this way coincide with the conformal gauge treatment (up to curvature terms at the one-dimensional boundary).

The principal question here is: What is the nature of δS ? In Ref.[17] it was noticed that the ξ -dependence represents parametrization ambiguities of the effective action (and hence must vanish on shell), although their specific realization was not clarified. The origin of this quantum reparametrization has nothing to do with gauge fixing because ξ had entered the action before a particular gauge was imposed. The problem is not likely to be resolved in the formalism of the Vilkovisky-DeWitt unique effective action [15]: its “uniqueness” does not rule out the possibility of having results that depend on the choice of the configuration-space metric, and the preferred metric (14) contains ξ explicitly.

Another related issue is that δS is identically zero at $d = 2$ since the Bianchi identities hold for arbitrarily large metric disturbances (i.e., $\tilde{R}_{\mu\nu} \equiv (1/2)\tilde{R}\tilde{g}_{\mu\nu}$ in our notations); then, how can one obtain non-trivial contributions just by expanding (1)? The question may be re-formulated as follows: What action corresponds to δS at the tree level? There seems to be no appropriate action functional at $d = 2$.

In view of the degeneracy alluded to above, the natural place to try is $2+\epsilon$ dimensions rather than exactly two: there must appear new contributions like $R_{\mu\nu}R^{\mu\nu}$ or $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$. (We simply ignore the latter aiming at the continuation to $d = 4$.) To keep up with the dimensions an arbitrary unit of mass μ should be introduced. Consider the following term

$$\mathcal{A} = 2\xi\mu^{2\epsilon} \int d^d x \sqrt{g} \omega \left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{2}R^2 \right). \quad (15)$$

If we expand (15) in powers of the quantum fields $\{h; \bar{h}_{\mu\nu}\}$ and *after this* take the limit $\epsilon \rightarrow 0$, then $\delta S^{(2)}$ is immediately reproduced. (It is important to realize that taking the limit $\epsilon \rightarrow 0$ does not commute with splitting the metric into its background and quantum parts.) There is no extra contribution

to the first order in fluctuations (i.e., $\mathcal{A}^{(1)} = 0$) and the equations of motion remain unmodified. In other words, one studies a theory in which there are $2 + \epsilon$ -dimensional quantum fluctuations around the purely two-dimensional background. Let us emphasize that the quantum reparametrization is equivalent to adding the term $\int d^2x \sqrt{g} R_{\mu\nu} R^{\mu\nu}$ into the initial action (2).

The limit $\xi \rightarrow \infty$ regains the corresponding one-loop determinant in the conformal gauge. This has become obvious by Eq.(15): as ξ goes to infinity the least action principle singles out the manifolds which obey (1), and segregates the fluctuations in the orthogonal ϵ dimensions. This situation is typical for theories which cannot be continued self-consistently beyond the number of dimensions they are defined in. Basically, the effect is not without physical significance: it is a clear witness of anomaly (see, e.g., an analogous discussion in the two-dimensional Wess-Zumino-Witten model, [18]).

3 Quantum Reparametrizations and Asymptotic Freedom in Dilaton Gravity

The same construction as discussed in the previous section, may be employed in a conventional (second-order in derivatives) dilaton gravity, which is a straightforward generalization of the Jackiw-Teitelboim model. With the help of conformal rescalings of the metric and general transformations of the scalar field, [19, 20], the action of the most general model may be always written in the form:

$$S = - \int d^2x \sqrt{g} \left[\frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + \Phi R + V(\Phi) \right] , \quad (16)$$

so that all the arbitrariness resides in the form of the potential function $V(\Phi)$.

The field transformations cannot remove a unique feature of the dilaton gravity: the direct dilaton-curvature coupling. However, the coefficients in (16) are subject to change. In particular, an apparent kinetic term, $(\partial\Phi)^2$,

may be set to zero by an appropriate conformal rescaling of the metric. This should not disturb us because it is the ΦR -term that carries the genuine (mixed) kinetic matrix for both the conformal mode and the dilaton. Diagonalizing this kinetic matrix one finds that the signs of the eigenmodes are opposite so that the model has zero dynamical degrees of freedom on shell, [20].

Within the background field treatment, the action (16) may be supplemented by, [17],

$$\delta S = \xi \int d^2x \sqrt{g} \Phi \bar{h}^{\mu\nu} \left[\tilde{R}_{\mu\nu} - \frac{1}{2} \tilde{R} \tilde{g}_{\mu\nu} \right] . \quad (17)$$

At one loop, the expression in square brackets should be expanded to the first order in quantum fluctuations $\{\varphi; h; \bar{h}_{\mu\nu}\}$. In the simplest covariant gauge,

$$\chi^\mu = \frac{1}{(1 + \xi)\Phi} \nabla^\mu \varphi - \nabla_\nu \bar{h}^{\mu\nu} , \quad (18)$$

$$\hat{C}_{\mu\nu} = -(1 + \xi)\Phi g_{\mu\nu} , \quad (19)$$

the correspondent determinant is easily done with the Schwinger-DeWitt technique yielding

$$\Gamma_{div} = -\frac{1}{4\pi\epsilon} \int d^2x \sqrt{g} \left[2R + V' + \frac{1}{(1 + \xi)\Phi} V - \frac{1}{(1 + \xi)\Phi} \Delta\Phi \right] , \quad (20)$$

modulo total derivatives of the dilaton. By using the equations of motion for the background fields,

$$\frac{1}{\sqrt{g}} \frac{\delta S}{\delta \Phi} \equiv \Delta\Phi - R - V' = 0 , \quad \frac{1}{\sqrt{g}} g_{\mu\nu} \frac{\delta S}{\delta g_{\mu\nu}} \equiv \Delta\Phi - V = 0 , \quad (21)$$

Γ_{div} can be put into the form, [11, 20, 17]:

$$\Gamma_{on\ shell} = \frac{1}{4\pi\epsilon} \int d^2x \sqrt{g} V' . \quad (22)$$

The standard renormalization routine gives the following class of ultraviolet-finite potentials:

$$V(\Phi) = b \exp(\alpha\Phi) , \quad (23)$$

where b and α are arbitrary constants. To understand the arbitrariness of the exponent α one notes that the second equation of motion (sometimes termed the “classical conformal anomaly”) amounts to a statement that the operator $V(\Phi)$ is a total derivative on shell, [17], thus one can add an arbitrary amount of $\int d^d x \sqrt{g} V$ to Eq.(22).

The reference to ξ has disappeared from the final on-shell expression, as it should be for a parametrization-dependent quantity. Moreover, an explicit evaluation of the determinant reveals that the intermediate expressions in the heat kernel expansion are regular functions of ξ as long as $\xi \neq -1$. Upon properly accounting for the contributions to the path-integral measure, all the configuration space matrices which enter the relevant Seeley-Gilkey coefficient, [8], remain bounded when ξ grows indefinitely. The limit $\xi \rightarrow \infty$ returns us to the conformal gauge, up to an on-shell surface term $(8\pi\epsilon)^{-1} \int d^d x \sqrt{g} V$. We use this circumstance below to fix the overall scale of the counterterms.

Again, we can re-formulate the source of the ξ -dependence introducing the following piece

$$\mathcal{A} = -2\xi\mu^{2\epsilon} \int d^d x \sqrt{g} \Phi R , \quad d = 2(1 + \epsilon) , \quad (24)$$

and studying $2 + \epsilon$ -dimensional disturbances of the two-dimensional background. However, such an approach has an important setback: \mathcal{A} is no more zero at $\epsilon = 0$. This crucial difference with the case of the R^2 -gravity has to do with the conformal structures of the pertinent terms in the both models. In fact, there can be no new geometrical term with the background dimensionality equal to d . If one insists on having the term \mathcal{A} as an addition to (16), then the first equation of motion changes to:

$$\frac{1}{\sqrt{g}} \frac{\delta(S + \mathcal{A})}{\delta\Phi} \equiv \Delta\Phi - (1 + 2\xi)R - V' = 0 , \quad (25)$$

while the equation for the classical conformal anomaly persists. The reduction of Γ_{div} with the help of Eq.(25) leads to penetration of the ξ -dependence

into the topological divergence, which reminds us of the presence of the conformal anomaly:

$$\Gamma_{modified} = \frac{1 - 2\xi}{4\pi\epsilon} \int d^2x \sqrt{g} R , \quad (26)$$

where we have flipped the sign of ϵ for consistency. Now, there appears a point in the parameter space, $\xi = 1/2$, where the on-shell divergences vanish. This fact may lead us to the following speculation.

As the inclusion of $\delta S^{(2)}$ only affects the $\bar{h}_{\mu\nu}\bar{h}_{\alpha\beta}$ -sector, the two ΦR terms, in (16) and in (24), may be thought of as describing in-the-surface and out-of-the-surface quantum fluctuations, respectively. When the effect of the latter becomes significant (i.e., well beyond two dimensions) one can expect that a non-trivial fixed point of the gravitational beta-function exists. This is already seen in the leading order of the ϵ -expansion. Replacing the quantity $25 - c$ with $24 - c$, [21, 20], where c is the Virasoro central charge for the conformal matter fields, in the pertinent expressions of Ref.[3], one finds the following gravitational beta-function ($c = 0$):

$$\beta(\kappa) \equiv \mu \frac{\partial \kappa}{\partial \mu} = \epsilon\kappa - \kappa^3/\pi . \quad (27)$$

In addition to the trivial fixed point $\kappa = 0$, there exists another one,

$$\kappa_c^2 = \pi\epsilon , \quad \beta(\kappa_c) = 0 , \quad \beta'(\kappa_c) < 0 . \quad (28)$$

The simplest way to see how this obtains in our setting is to take $\xi \rightarrow \infty$, which specifies the rigid scale⁵ in the underlying $d = 2$ theory, and to freeze the dilaton field at its natural value $\langle \Phi \rangle = 1/(2\kappa^2)$. Then we have

$$\frac{1}{2\kappa_0^2} = \mu^{2\epsilon} \left(\frac{1}{2\kappa^2} - \frac{1}{2\pi\epsilon} \right) , \quad (29)$$

where κ_0 is the bare Einstein constant, which leads directly to Eq.(27).

⁵This procedure is mandatory as emphasized in the first paper of Ref.[3] because there is no default scale in the pure Einstein action.

Dropped from Eq.(26) are the convergent terms: those contribute $\mathcal{O}(\kappa^3\epsilon)$ to the beta-function (27). As the loop expansion is justified for small values of κ , the omitted contributions are indeed negligible whilst the two remaining terms may be on the same order of magnitude, $\kappa^2 \simeq \epsilon$. The latter is manifest for the ultraviolet stable point κ_c .

4 Conclusions

In this paper, we have studied quantum reparametrizations in two-dimensional models of gravity. We have pursued the point of view that a dimensional extension $d = 2n \rightarrow 2(n + \epsilon)$ of a geometrical theory is encoded in the parametrization structure of the model formulated in the basic number $2n$ of dimensions. This has been demonstrated on a comparatively simple example of the two-dimensional R^2 -gravity. The related discussion for the (conventional) dilaton gravity has also been presented.

We have provided some support for a popular speculation that the strong-coupling gravity belongs to a different universality class from its weak coupling version. In the dilaton gravity, we have found an ultraviolet fixed point for the Einstein constant flow to the leading order in the ϵ -expansion around two dimensions. It would be interesting to couple matter fields to the action (16) and to find the relevant operators at the fixed point κ_c . We must admit, however, that although the dilaton gravity is as legitimate as the Einstein theory at $d = 2$ they definitely have different behaviors in higher dimensions.

Technically, we have shown how to proceed with the Schwinger-DeWitt technique in the two-dimensional R^2 -gravity where the highest-derivative term in the one-loop determinant is degenerate within the linear metric parametrization and no operator squaring [8] can help. The construction is straightforwardly generalized to a much more complicated case of a scalar-tensor R^2 -gravity [16]. We believe that our approach may be useful in other models as well.

Our final remark concerns another possible interpretation of Eq.(17). Note that the background field Φ is an artifact of a specific realization: it simply sets the scale. Contrary to that, the “graviton” field $\bar{h}_{\mu\nu}$ may be viewed as a Lagrange multiplier that enforces the two-dimensional Bianchi identities. When Eq.(1) holds the Lagrange multiplier is unimportant; conversely, when the auxiliary field is introduced the constraint (1) may be relaxed. This observation might be the first step towards the dual description of low-dimensional gravity.

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